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RESEARCH MEMORANDUM

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A QUADRATICALLY CONVERGENT PARALLEL JACOBI PROCESS FOR ALMOST DIAGONAL MATRICES WITH DISTINCT EIGENVALUES

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A Quadratically Convergent Parallel Jacobi Process for Almost Diagonal Matrices with Distinct Eigenvalues.

Abstract

This paper discusses a generalization of the Jacobi process (1846). In each step $\frac{1}{2}$ n pairs of non-diagonal elements are annihilated in an arbitrary almost diagonal matrix. We proof that the recursively constructed sequence of matrices converges to a diagonal matrix. The quadratically convergent method is adapted to parallel computation on an array processor.

1. INTRODUCTION

In the welknown Jacobi algorithm [5] for the diagonalization of a real *symmetric* matrix each of the successive *orthogonal* similarity transformations is designed to annihilate a symmetric pair of off-diagonal elements. The ultimately quadratic convergence of this algorithm has been investigated by several authors [6,7,9,12].

In this paper we describe a natural, despite new generalization of that Jacobi process for the diagonalization of a *non-normal* almost diagonal matrix $A \in \mathbb{C}^{n\times n}$ with distinct eigenvalues. We assume n to be even. In each iteration

$$A^{(k+1)} = S_k^{-1} A^{(k)} S_k, \; k \in \mathbf{Z} , \tag{1.1}$$

with $A^{(0)} := A$, $\frac{1}{2}$ n pairs of symmetrically placed off-diagonal elements are annihilated. The transformation matrices S_k are not unitary. Consequently monotonic decrease of the Frobenius norms of the non-diagonal parts of $A^{(k)}$ can not be guaranteed despite the annihilation of elements.

In [9] the same lack of monotonicity caused genuine difficulties in the proof of the quadratic convergence of the Eberlein algorithm [3,8] for the algebraïc eigenproblem.

Each S_k is a direct sum of $\frac{1}{2}$ n unimodular shears $T_{i,k}$, i = 1,..., n/2;

$$T_{i,k} = \begin{array}{cc} \begin{pmatrix} p_{i,k} & q_{i,k} \\ t_{i,k} & s_{i,k} \end{pmatrix} & \begin{array}{l} \leftarrow \ell(i,k) \\ \leftarrow m(i,k) \end{array} \\ \begin{array}{cc} \uparrow & \uparrow \\ \ell(i,k) & m(i,k) \end{array} & \end{array} \tag{1.2}$$

They annihilate $\frac{1}{2}$ n pairs of the transformed matrix. In case of hermitian A each $T_{i,k}$ is a unitary 2 × 2 block. Our strategy for the pivots ($\ell(i,k)$, $m(i,k)$) aims to annihilate each off diagonall element exactly once in n-1 successive steps:

$$\{(\ell(i,k),m(i,k))\,|\,i=1,\dots,n/2,k=0,\dots,n-2\} = \{i,j)\,|\,1\leqq i<j\leqq n\}. \tag{1.3}$$

If the difference of $A^{(k)} = D^{(k)} + E^{(k)}$ and its diagonal part $D^{(k)} = \mathrm{diag}(a_{jj}^{(k)})$ is small relatively the separation of the eigenvalues, the transformation matrix S_k almost equals the unit matrix. Then the previously annihilated elements can increase only slowly. That phenomenon gives rise to the quadratic convergence:

$$\|E^{(n-1)}\|_F \leqq \text{constant } \|E^{(0)}\|_F^2 . \tag{1.4}$$

The ideas of Brent and Luk [2] for the solution of the symmetric eigenproblem on a systolic array can be carried over to our method. Further may be remarked that our procedure can be used as final stage in Sameh's (slowly convergent) parallel normreducing eigenvalue algoritm [10], in order to obtain a quadratically convergent process.
The pivots of T_{ik} are $(\ell(i,k), m(i,k))$, $i = 1,\dots,n/2$, with $\ell(i,k)$ $m(i,k)$ and $\ell(i,k) < \ell(j,k)$ for $i < j$. (1.5)

The caterpillar permutation K_n[2] generates a pivot strategy that annihilates each off-diagonal element exactly once in a _sweep_ of $\frac{1}{2}n(n-1)$ shear transformations. These are performed in n-1 _steps_. Permutation K_n can be read of from figure 1.

$$\begin{array}{ccccccccccc}
1 & 3 & \rightarrow & 5 & \rightarrow & \dots & \rightarrow & n-3 & \rightarrow & n-1 \\
 & & & & & & & & & \downarrow \\
2 \leftarrow & 4 & \leftarrow & 6 & \leftarrow & \dots & \leftarrow & n-2 & \leftarrow & n
\end{array}$$

fig. 1 Caterpillar permutation K_n.

If $b^{(0)} = (1,2,\dots,n)^T$, then

$$(K_n b^{(0)})_i = \begin{cases} i+2 & , \quad 2 \leqq i = 2k \leqq n-2 \\ i-2 & , \quad 5 \leqq i = 2k-1 \leqq n-1 \\ i-1 & , \quad i = 3,n \\ i & , \quad i = 1 . \end{cases} \tag{1.6}$$

With $b^{(k)} = K_n^k \, b^{(0)}$ we have

$$\{\ell(i,k),\ m(i,k)\} = \{b_{2i-1}^{(k)},\ b_{2i}^{(k)}\}\ ,\ i = 1,\ldots,\ \frac{1}{2}\ n. \qquad (1.7)$$

Figure 2 gives the pivot pairs $(\ell(i,k),m(i,k))$ in the catherpillar order for $n = 6$

k \ i	1	2	3
0	(1,2)	(3,4)	(5,6)
1	(1,4)	(2,6)	(3,5)
2	(1,6)	(2,3)	(4,5)
3	(1,5)	(2,4)	(3,6)
4	(1,3)	(2,5)	(4,6)

fig. 2

For the proof of the quadratic convergence of the cyclic generalized Jacobi-process the rule (1.3) is essential.
For the description of the process we divide $A^{(k)}$ into a diagonal and a nondiagonal part $D^{(k)}$ and $E^{(k)}$ respectively.

$$E^{(k)} = A^{(k)} - D^{(k)},\ d_{ij}^{(k)} = \begin{cases} a_{ij}^{(k)} & ,\ i = j \\ 0 & ,\ i \neq j. \end{cases} \qquad (1.8)$$

We set

$$\varepsilon_k = \| E^{(k)} \|_F \qquad (1.9)$$

and

$$\vartheta_k = \min_{i \neq j} |a_{ii}^{(k)} - a_{jj}^{(k)}|. \qquad (1.10)$$

The initial matrix A is almost diagonal namely

$$\varepsilon_0 \leqq \eta/10, \qquad (1.11)$$

where η denotes the separation of the spectrum of A:

$$\eta = \min_{i \neq j} |\lambda_i - \lambda_j|.$$

Condition (1.11) implies [1] that each eigenvalue λ_i is affiliated to exactly one diagonal element, say a_{ii}

$$|\lambda_i - a_{ii}| \leqq \varepsilon_0 \tag{1.12}$$

Consequently $|a_{ii} - a_{jj}| \geqq \eta - 2\,\varepsilon_0 \geqq 8\,\varepsilon_0$. Thus

$$\varepsilon_0 \leqq \vartheta_0/8. \tag{1.13}$$

The organisation of our paper is as follows. In section 2 we present the parallel annihilators together with their properties and effects during the first step. In section 3 we describe the consequences of the annihilations in a complete sweep. In that section the main result is attained:

if $n \geqq 7$ and $\varepsilon_0/\eta \leq (10n)^{-1}$ then $\varepsilon_{n-1}/\eta \leqq 2n(\varepsilon_0/\eta)^2 \leqq \frac{1}{5}\,\varepsilon_0/n$.

Section 4 discusses conclusions and some unsolved problems.

2. PARALLEL ANNIHILATORS. THE FIRST STEP

Without loss of generality we may assume that

$$(\ell(i,0),\ m(i,0)) = (2i-1,2i),\ i = 1,\dots,\ \frac{1}{2}\, n$$

as in the caterpillar strategy. Now we consider the annihilation in the first step, $k = 0$, with the annihilators $T_{i,0}$. For reasons of simplicity the subscript $k = 0$, referring to the first step will be omitted in the formulae of this section.

The shear transformation T_i only affects the elements in the $(2i-1)$-th and the $2i$-th row and column.

The Jacobi-parameters of T_i are p_i, q_i, r_i, s_i:

$$T_i = \begin{bmatrix} p_i & q_i \\ r_i & s_i \end{bmatrix},\ p_i s_i - q_i r_i = 1,\ i = 1,\dots,\ \frac{1}{2}\, n. \tag{2.1}$$

Let matrix A be partitioned as

$$A = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{1,\frac{1}{2}n} \\ A_{21} & A_{22} & \cdots & A_{2,\frac{1}{2}n} \\ \vdots & \vdots & & \vdots \\ A_{\frac{1}{2}n,1} & A_{\frac{1}{2}n,2} & \cdots & A_{\frac{1}{2}n,\frac{1}{2}n} \end{bmatrix} = (A_{ij}), \tag{2.2}$$

where each 2×2 submatrix A_{ij} is given by

$$A_{ij} = \begin{bmatrix} a_{2i-1,2j-1} & a_{2i-1,2j} \\ a_{2i,2j-1} & a_{2i,2j} \end{bmatrix},\ i,j = 1,\dots,\ \frac{1}{2}\, n. \tag{2.3}$$

For the 2×2 blocks $A_{ij}^{(1)}$ of $A^{(1)} = S^{-1}AS$, where $S = \mathrm{diag}(T_i)$, we find

$$A_{ij}^{(1)} = T_i^{-1} \otimes T_j^T A_{ij}\ . \tag{2.4}$$

Especially for the diagonal block $A_{ii}^{(1)}$:

$$\begin{bmatrix} a_{2i-1,2i-1}^{(1)} \\ a_{2i-1,2i}^{(1)} \\ a_{2i,2i-1}^{(1)} \\ a_{2i,2i}^{(1)} \end{bmatrix} = \begin{bmatrix} s_i p_i & s_i r_i & -q_i p_i & -q_i r_i \\ s_i q_i & s_i^2 & -q_i^2 & -q_i s_i \\ -r_i p_i & -r_i^2 & p_i^2 & p_i r_i \\ -r_i q_i & -r_i s_i & p_i q_i & p_i s_i \end{bmatrix} \begin{bmatrix} a_{2i-1,2i-1} \\ a_{2i-1,2i} \\ a_{2i,2i-1} \\ a_{2i,2i} \end{bmatrix} . \quad (2.5)$$

In order to simplify the formulae we set

$$\sigma_i = a_{2i,2i-1} \, , \; \mu_i = a_{2i-1,2i} \, , \; \nu_i = a_{2i-1,2i-1} - a_{2i,2i} , \; i=1,\dots,\tfrac{1}{2} n. \quad (2.6)$$

The annihilation of both σ_i and μ_i requires the solution of a quadratic equation with discriminant $\nu_i^2 + 4\,\sigma_i \mu_i$.
With (1.13) we get $|2\sigma_i \mu_i| \leqq \varepsilon^2 \leqq \vartheta^2/64 \leqq |\nu_i|^2/64$. Thus

$$|1 + 4\sigma_i \mu_i \, \nu_i^{-2}| \geqq 31/32 \, . \quad (2.7)$$

Let be

$$F_i := (1+4\,\sigma_i \mu_i \nu_i^{-2})^{\frac{1}{2}} \, , \; \mathrm{Re}(F_i) > 0, \; 1 \leqq i \leqq \tfrac{1}{2} n.$$

Lowerbound (2.7) enables the formulation of

THEOREM 2.1 The shears

$$T_i = \begin{bmatrix} p_i & q_i \\ r_i & s_i \end{bmatrix} \begin{bmatrix} \left(\frac{1}{2} + \frac{1}{2F_i}\right)^{\frac{1}{2}} & \dfrac{-\mu_i}{\nu_i (F_i(1+F_i)/2)^{\frac{1}{2}}} \\ \dfrac{\sigma_i}{\nu_i (F_i(1+F_i)/2)^{\frac{1}{2}}} & \left(\frac{1}{2} + \frac{1}{2F_i}\right)^{\frac{1}{2}} \end{bmatrix} , \; i=1,\dots,n/2 \quad (2.8)$$

annihilate both σ_i and μ_i. Such a shear T_i is called the *annihilator*, related to the pivots in question.

PROOF. With (2.11) we derive that $a^{(1)}_{2i,2i-1} = 0$ and $a^{(1)}_{2i-1.2i} = 0$ if

$$\begin{pmatrix} p_i \\ q_i \end{pmatrix} = \alpha_i \begin{pmatrix} (1+F_i)/2 \\ \sigma_i \nu_i^{-1} \end{pmatrix} , \quad \begin{pmatrix} r_i \\ s_i \end{pmatrix} = \beta_i \begin{pmatrix} -\mu_i \nu_i^{-1} \\ (1+F_i)/2 \end{pmatrix} , \quad \alpha_i, \beta_i \in \mathbb{C}\setminus\{0\}.$$

The condition $p_i s_i - q_i r_i = 1$ implies $\alpha_i \beta_i = 2(F_i + F_i^2)^{-1}$. With $\alpha_i = \beta_i = (2/(F_i+F_i^2))^{\frac{1}{2}}$ we get (2.15). □

In the sequel we need the estimates in

THEOREM 2.2. If $\varepsilon \leqq \eta/10$ then the following inequalities hold for T_i, $i = 1,\ldots,\frac{1}{2}n$.

(a) $|p_i|^2 \leqq 1 + (\varepsilon/\eta)^2 \leqq 1 + (\varepsilon/\vartheta)^2$; (2.9)

(b) $|F_i + F_i^2|^{-1} \leqq \frac{1}{2} + \frac{5}{4}\varepsilon^2/\eta^2$; (2.10)

(c) $\|T_i\|_2^2 \leqq 1 + 2\,\varepsilon/\eta \leqq \frac{6}{5}$. (2.11)

(d) $|(F_i+F_i^2)/2|^{-\frac{1}{2}} \leqq 1 + \frac{6}{5}\varepsilon^2/\eta^2$. (2.12)

PROOF (a). $|p_i|^2 = |\ \frac{1}{2} + \frac{1}{2}\,(1+4\,\sigma_i\nu_i^{-2})^{-\frac{1}{2}}\ | \leqq \frac{1}{2} + \frac{1}{2}\,(1-2\,\varepsilon^2/\vartheta^2)^{-\frac{1}{2}}/\eta)^{-2}$

$$\leqq \frac{1}{2} + \frac{1}{2}\,(1-2(\varepsilon/\eta)^2(1-2\varepsilon/\eta)^{-2})^{-\frac{1}{2}}$$

since $\vartheta \geqq \eta-2\varepsilon$. If $|x| < 1/10$ then $(1-2\,x^2(1-2x)^{-2})^{-\frac{1}{2}} \leqq 1 + 2x^2$.

Hence $|p_i|^2 \leqq 1 + \varepsilon^2/\eta^2 \leqq 1 + \varepsilon^2/\vartheta^2$.

(b) $|F_i + F_i^2|^{-1} = |1+4\,\sigma_i\,\mu_i\nu_i^{-2} + (1+4\,\sigma_i\mu_i\nu_i^{-2})^{\frac{1}{2}}|^{-1}$

$$\leqq (1-2x^2(1-2x)^{-2} + (1-2x^2(1-2x)^{-2})^{\frac{1}{2}})^{-1}$$

where as above $x = \varepsilon/\eta$. Simple but tedious calculations affirm

$$(1-2x^2(1-2x)^{-2} + (1-2x^2)(1-2x)^{-2})^{\frac{1}{2}})^{-1} \leqq \frac{1}{2} + \frac{5}{4}x^2, \ |x| \leqq 1/10.$$

(c) Let be $x_i = |p_i|^2 + |q_i|^2$, $y_i = |r_i|^2 + |s_i|^2$, $z = p\bar{r}_i + q\bar{s}_i$. Then

$$T_iT_i^* = \begin{bmatrix} x_i & z_i \\ \bar{z}_i & y_i \end{bmatrix}$$

and $x_iy_i - |z_i|^2 = 1$ as follows from the unimodularity of annihilator T_i.

So $x_i + y_i = ((x_i-y_i)^2 + 4|z_i|^2 + 4)^{\frac{1}{2}} \geqq 2$ and

$$\|T_i\|_2^2 = (x_i+y_i+((x_i+y_i)^2 - 4)^{\frac{1}{2}})/2 \tag{2.13}$$

With (2.8) we get

$$\begin{aligned} x_i + y_i - 2 &= 2(|\tfrac{1}{2} + \tfrac{1}{2}F_i^{-1}|-1) + 2(|\sigma_i)^2 + |\mu_i|^2)/|\nu_i^2(F_i+F_i^2)| \\ &\leqq |(1-F_i)/F_i| + 2\varepsilon^2/|\nu_i^2(F_i+F_i^2)| \\ &\leqq (4|\sigma_i\mu_i| + 2\varepsilon^2)/|\nu_i^2(F_i+F_i^2)| \leqq (2+5\varepsilon^2/\eta^2)\varepsilon^2/\vartheta^2 \end{aligned}$$

as follows from (2.10). Hence

$$\|T_i\|_F^2 = x_i + y_i \leqq 2 + 2.05\ \varepsilon^2/\vartheta^2 \tag{2.14}$$

From (2.13) and (2.14) it follows with easy calculations that

$$\|T_i\|_2^2 \leqq 1 + \frac{8}{5}\varepsilon/\vartheta \leqq 1 + \frac{8}{5}\varepsilon\eta^{-1}(1-2\varepsilon/\eta)^{-1} \leqq 1 + 2\ \varepsilon/\eta \leqq \frac{6}{5}$$

for $\varepsilon \leqq \eta/10$.

(d) The derivation of (2.12) is similar to that of (2.10). □

An upperbound for the growth of norm of the nondiagonal part after the first step is given in

THEOREM 2.3. If $\varepsilon < \eta/10$, then

$$\varepsilon_1 \leqq (1+2\varepsilon/\eta)\ \varepsilon \leqq \frac{6}{5}\ \varepsilon. \tag{2.15}$$

PROOF. The 2×2 blocks in the partitioned $A^{(1)}$ are

$$A_{ij}^{(1)} = T_i^{-1} A_{ij} T_j \quad , i,\ j = 1, \dots, \frac{n}{2}.$$

Since $a_{2i-1,2i}^{(1)} = a_{2i,2i-1}^{(1)} = 0$, $i = 1, \dots, \frac{n}{2}$, with (2.11) we find

$$\varepsilon_1^2 = \sum_{i \neq j} \|A_{ij}^{(1)}\|_F^2 \leqq \sum_{i \neq j} \|T_i^{-1}\|_2^2\ \|T_j\|_2^2\ \|A_{ij}\|_F^2$$

$$\leqq \sum_{i \neq j} \|T_i\|_2^2\ \|T_j\|_2^2\ \|A_{ij}\|_F^2 \leqq (1+2\varepsilon/\eta)^2\ \varepsilon^2. \qquad \square$$

According to (1.12) $a_{ii}^{(0)}$ is affiliated with λ_i and

$$|a_{ii}^{(0)} - \lambda_i| \leqq \varepsilon_0.$$

This affiliation is not changed by the annihilating transformation. This will be proved in

THEOREM 2.4. If $\varepsilon < \eta/10$ then the affiliation of the eigenvalues remains unchanged by the annihilating transformation

$$A^{(1)} = S^{-1}\ A\ S$$

where $S = \mathrm{diag}(T_i)$ with T_i given by (2.8)

PROOF. From (2.5) and (2.8) one gets

$$|a_{2i-1,2i-1}^{(1)} - a_{2i-1,2i-1}| = |2\ \sigma_i \mu_i \nu_i^{-1} (1+F_i)^{-1}|$$

$$\leqq \varepsilon^2 |\nu_i|^{-2}\ (1+(1-2\varepsilon^2|\nu_i|^{-2})^{\frac{1}{2}})^{-1}\ |\nu_i|$$

$$\leqq (\varepsilon/\vartheta)^2 (1+(1-2(\varepsilon/\vartheta)^2)^{\frac{1}{2}})^{-1}\ |\nu_i| \leqq |\nu_i|/126, \tag{2.16}$$

for $\varepsilon/\vartheta \leqq 1/8$. A change of affiliation only can be an exchange of affiliation between two elements at say the i-th shear transformation. Assume $a_{2i-1,2i-1}$ and $a^{(1)}_{2i-1,2i-1}$ to be affiliated with λ_{2i-1} and λ_{2i} respectively. Then as follows from (2.15)

$$|a_{2i-1,2i-1} - \lambda_{2i-1}| \leqq \varepsilon \quad , \; |a^{(1)}_{2i-1,2i-1} - \lambda_{2i}| \leqq \frac{6}{5}\,\varepsilon .$$

By (2.16)

$$|a^{(1)}_{2i-1,2i-1} - a_{2i-1,2i-1}| \leqq \frac{1}{126}\,|\nu_i|$$

$$\leqq \frac{1}{126}\,(|\lambda_{2i-1}-\lambda_{2i}| + 2\varepsilon). \tag{2.17}$$

The change of affiliation implies

$$|a^{(1)}_{2i-1,2i-1} - a_{2i-1,2i-1}| \geqq |\lambda_{2i-1} - \lambda_{2i}| - \frac{11}{5}\,\varepsilon . \tag{2.18}$$

Consequently we get with (2.17) and (2.18)

$$|\lambda_{2i-1} - \lambda_{2i}| - \frac{11}{5}\,\varepsilon \leqq \frac{1}{126}|\,\lambda_{2i-1} - \lambda_{2i}| + \frac{1}{63}\,\varepsilon$$

Thus $\varepsilon > \frac{625}{1396}\,|\lambda_{2i-1} - \lambda_i| \geqq \frac{2}{5}\,\eta$. This contradicts (1.11). □

3. THE EFFECT OF A COMPLETE SWEEP

For the estimates of the norm ε_k of the nondiagonal part $E^{(k)}$ of $A^{(k)}$, $k = 0,\dots, n-1$, we make use of the conclusion based on theorem 2.3: if $\varepsilon_k/\eta \leqq \frac{1}{10}$ then

$$\varepsilon_{k+1} \leqq (1+2\varepsilon_k/\eta)\varepsilon_k. \tag{3.1}$$

In the first theorem we proof that a sufficient small ε_0 guarantees a slow growth of ε_k.

THEOREM 3.1. If $\varepsilon_0 \leqq \frac{1}{10}\eta\, n^{-1}$ then

$$\varepsilon_k \leqq \frac{1}{10}\eta(n-k)^{-1},\ k = 0,\dots, n-1. \tag{3.2}$$

PROOF. Assume $\varepsilon_k/\eta \leqq \frac{1}{10}(\eta-k)^{-1}$. Then, as follows from (3.1)

$$\varepsilon_{k+1}\eta^{-1} \leqq (1+2\varepsilon_k\eta^{-1})\ \varepsilon_k\eta^{-1} \leqq \frac{1}{10}(n-k)^{-1}\ (1+\frac{1}{5}(n-k)^{-1}).$$

It is easy to verify that

$$\frac{1}{10}(n-k)^{-1}\ (1+\frac{1}{5}(n-k)^{-1}) \leqq \frac{1}{10}(n-k-1)^{-1}. \qquad \square$$

The transform $A^{(k+1)}$ is obtained from $A^{(k)}$ in two stages:

$$\begin{cases} A^{(k+\frac12)} = S_k^{-1}A^{(k)} = H^{(k)}A^{(k)} + B^{(k)}, & (3.3)\\ A^{(k+1)} = A^{(k+\frac12)}S_k = A^{(k+\frac12)}H^{(k)} + C^{(k)} & (3.4)\end{cases}$$

where $H^{(k)}$ denotes the diagonal part of both S_k and S_k^{-1}.
Thus by (1.2)

$$\begin{cases} b^{(k)}_{\ell(i,k),j} = -q_{i,k}\,a^{(k)}_{m(i,k),j},\ b^{(k)}_{m(i,k),j} = -r_{i,k}\,a^{(k)}_{\ell(i,k),j},\ i = 1,\dots,n/2, & (3.5)\\ c^{(k)}_{j,\ell(i,k)} = r_{i,k}\,a^{(k+\frac12)}_{j,m(i,k)}, c_{j,m(i,k)} = q_{i,k} = q_{i,k}\,a^{(k+\frac12)}_{j,\ell(i,k)}, j = 1,\dots,n. & (3.6)\end{cases}$$

Let be

$$G^{(k)} = B^{(k)}H^{(k)} + C^{(k)} . \tag{3.7}$$

Then

$$A^{(k+1)} = H^{(k)}A^{(k)}H^{(k)} + G^{(k)} . \tag{3.8}$$

According to (3.8) the updating $A^{(k)} \to A^{(k+1)}$ comes about by two actions: a multiplication by two diagonal elements of S_k followed by the addition of $g_{ij}^{(k)}$. The multipliers $H^{(k)}$ are rather tame as can be seen in

THEOREM 3.2. If $\varepsilon_0 \leqq \frac{1}{10}\,\eta/n$ then

$$\prod_{k=2}^{n-2} \|H^{(k)}\|_2^2 \leqq 155/154 .$$

PROOF. With (2.9) and (3.2), we get

$$\prod_{k=2}^{n-2} \|H^{(k)}\|_2^2 \leqq \prod_{k=2}^{n-2} (1+\varepsilon_k^2/\eta^2) \leqq \prod_{k=2}^{n-2} (1+(n-k)^{-2}/100)$$

$$\leqq \prod_{k=2}^{\infty} (1+k^{-2}/100) \leqq 155/154$$

for $\quad \frac{1}{100} \sum_{k=2}^{\infty} k^{-2} = (\pi^2/6-1)/100$ and $\exp\ (\pi^2/6-1)/100) \leqq 155/154$. □

The twofold action (3.8) for the construction of $A^{(k+1)}$ is the starting point for the description on the history of the element $a_{ij}^{(n-1)}$, $i \neq j$, after its annihilation in some step. Let the element in the position (i,j) be annihilated in step $N(i,j) -1 = N-1$. Then

$$a_{i,j}^{(N)} = 0 \quad , \quad a_{i,j}^{(N+1)} = g_{i,j}^{(N)} \quad , \ i \neq j. \tag{3.9}$$

and

$$a_{i,j}^{(k+1)} = h_{i,i}^{(k)} \ h_{j,j}^{(k)} \ a_{i,j}^{(k)} + g_{i,j}^{(k)}, \quad N + 1 \leqq k \leqq n-2. \tag{3.10}$$

So

$$|a_{i,j}^{(k+1)}| \leqq \|H^{(k)}\|_2^2 \; |a_{i,j}^{(k)}| + |g_{i,j}^{(k)}|$$

and consequently

$$|a_{i,j}^{(n-1)}| \leqq \sum_{k=N}^{n-2} \prod_{m=k+1}^{n-2} \|H^{(m)}\|_2^2 \; |g_{i,j}^{(k)}|$$

$$\leqq \frac{155}{154} \sum_{k=N}^{n-2} |g_{ij}^{(k)}| \quad , \; i \neq j \; , \tag{3.11}$$

as follows from theorem 3.1.

The notational problems in the analysis of the parallel cyclic Jacobi-process are reduced with

DEFINITION 3.3. Let be $V = (v_{ij}) \in \mathbb{C}^{n\times n}$. Then $\hat{V}^{(k)} = (\hat{v}_{ij}^{(k)})$ and $\tilde{V}^{(k)} = (\tilde{v}_{ij}^{(k)})$, $k = 0,\dots, n-2$, are defined by

$$\hat{v}_{i,j}^{(k)} = \begin{cases} 0 & , \; i = j \text{ or } \{i,j\} \in \{\{\ell(i,k),m(i,k)\} \mid 0 \leqq i \leqq n/2\} \\ v_{i,j} & , \; (i,j) \text{ otherwise,} \end{cases} \tag{3.12}$$

$$\tilde{v}_{i,j}^{(k)} = \begin{cases} v_{i,j} & , \; i \neq j \wedge k > N(i,j) \\ 0 & , \; \text{otherwise,} \end{cases} \tag{3.13}$$

$$\delta_k(V) := \|\hat{V}^{(k)}\|_F \; . \tag{3.14}$$

□

Figure 3 illustrates the pattern of zeros in the successive matrices $\tilde{G}^{(k)}$, $k=1,\dots, n-2 = 4$. The □ indicates the position of an element annihilated in a preceding step. The corresponding elements in $\tilde{G}^{(k)}$ contribute according (3.8) to $a_{ij}^{(n-1)}$.

$$\tilde{G}^{(1)} = \begin{bmatrix} 0 & \square & 0 & 0 & 0 & 0 \\ \square & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \square & 0 & 0 \\ 0 & 0 & \square & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \square \\ 0 & 0 & 0 & 0 & \square & 0 \end{bmatrix} ; \tilde{G}^{(2)} = \begin{bmatrix} 0 & \square & 0 & \square & 0 & 0 \\ \square & 0 & 0 & 0 & 0 & \square \\ 0 & 0 & 0 & \square & \square & 0 \\ \square & 0 & \square & 0 & 0 & 0 \\ 0 & 0 & \square & 0 & 0 & \square \\ 0 & \square & 0 & 0 & \square & 0 \end{bmatrix} ;$$

$$\tilde{G}^{(3)} = \begin{bmatrix} 0 & \square & 0 & \square & 0 & \square \\ \square & 0 & \square & 0 & 0 & \square \\ 0 & \square & 0 & \square & \square & 0 \\ \square & 0 & \square & 0 & \square & 0 \\ 0 & 0 & \square & \square & 0 & \square \\ \square & \square & 0 & & \square & 0 \end{bmatrix} ; \tilde{G}^{(4)} \begin{bmatrix} 0 & \square & \square & \square & 0 & \square \\ \square & 0 & \square & 0 & \square & \square \\ \square & \square & 0 & \square & \square & 0 \\ 0 & \square & \square & \square & 0 & \square \\ 0 & \square & \square & \square & 0 & \square \\ \square & \square & 0 & \square & \square & 0 \end{bmatrix}$$

Fig. 3. The matrices $\tilde{G}^{k}$, $k=1,\dots, n-2 = 4$, in case of the caterpillar cyclic order.

Evidently $|\tilde{G}^{(k)}| \leqq \hat{G}^{(k)}|$. Our notation allows to reformulate (3.11) in matricial form:

$$|E^{(n-1)}| \leqq \frac{155}{154} \sum_{k=1}^{n-2} |\tilde{G}^{(k)}| \leqq \frac{155}{154} \sum_{k=1}^{n-2} |\hat{G}^{(k)}| .$$

Thus

$$\varepsilon_{n-1} \leqq \frac{155}{154} \sum_{k=1}^{n-2} \|\hat{G}^{(k)}\|_F . \tag{3.15}$$

In the proof of the quadratic convergence an essential role has been reserved for

THEOREM 3.4. Let be $G^{(k)}$ as defined in (3.7). If $\varepsilon_0/\eta \leqq (10n)^{-1}$, then

$$\|\hat{G}^{(k)}\|_F \leqq \frac{84}{83} \left(1 + \frac{7}{10} (n-k)^{-1}\right) \varepsilon_k^2/\eta \tag{3.16}$$

PROOF. Remark that $\|\hat{G}^{(k)}\|_F \leqq \|P^{(k)}\|_2 \, \|\hat{B}^{(k)}\|_F + \|\hat{C}^{(k)}\|_F$. Firstly we estimate $\|\hat{C}^{(k)}\|_F$. By (3.6) and (3.12)

$$\|\hat{C}^{(k)}\|_F \leqq \sum_{i=1}^{n/2} \Big(\big(\sum_{j=1}^{n}{}^{(i)} |c^{(k)}_{j,\ell(i,k)}|^2 \big)^{\frac{1}{2}} + \big(\sum_{j=1}^{n}{}^{(i)} |c^{(k)}_{j,m(i,k)}|^2 \big)^{\frac{1}{2}} \Big)$$

$$\leqq \sum_{i=1}^{n/2} \Big(|r_{i,k}| \big(\sum_{j=1}^{n}{}^{(i)} |a^{(k+\frac{1}{2})}_{j,m(i,k)}|^2 \big)^{\frac{1}{2}} + |q_{i,k}| \big(\sum_{j=1}^{n}{}^{(i)} |a^{(k+\frac{1}{2})}_{j,\ell(i,k)}|^2 \big)^{\frac{1}{2}} \Big) .$$

where $\sum_{j=1}^{n}{}^{(i)}$ denotes summation for all j except $j = \ell(i,k),\ m(i,k)$.
With (2.8) it follows that

$$\|\hat{C}^{(k)}\|_F \leqq \sum_{i=1}^{n/2} \rho_{i,k} \Big(|\sigma_{i,k}| \big(\sum_{j=1}^{n}{}^{(i)} |a^{(k+\frac{1}{2})}_{j,m(i,k)})^2 \big)^{\frac{1}{2}} + |\mu_{i,k}| \big(\sum_{j=1}^{n}{}^{(i)} |a^{(k+\frac{1}{2})}_{j,\ell(i,k)}|^2 \big)^{\frac{1}{2}} \Big)$$

where $\sigma_{i,k} = a^{(k)}_{m(i,k),\ell(i,k)}$, $\mu_{i,k} = a^{(k)}_{\ell(i,k),m(i,k)}$ and

$\rho_{i,k} = |\nu^2_{i,k}(F_{i,k}+F^2_{i,k})^2|^{-\frac{1}{2}}$.

Now $|\nu_{i,k}|^{-1} \leqq \eta^{-1}(1-2\ \varepsilon_k/\eta)^{-1} \leqq \eta^{-1}(1-(n-k)^{-1}/5)^{-1} \leqq (1+(n-k)^{-1}/4)/\eta$ for

$n - k \geqq 2$. (2.12) and (3.2) imply $|(F_{i,k}+F^2_{i,k})/2|^{-\frac{1}{2}} \leqq 1 + \frac{6}{5}\ \varepsilon^2_k/\eta^2 \leqq 1.012$.

Hence

$$\|\hat{C}^{(k)}\|_F \leqq 0.506(1+(n-k)^{-1}/4)\ (\delta^2_k(A^{(k+\frac{1}{2})} + \sum_{i=2}^{n/2} (|\sigma_{ik}|^2 + |\mu_{ik})^2)/\eta$$

$$\leqq \frac{42}{83}\ (1+(n-k)^{-1}/4)\ \|S_k\|^2_2\ \varepsilon^2_k/\eta \leqq \frac{42}{83}\ (1+(n-k)^{-1}/4)\ (1+(n-k)^{-1}/5)\ \varepsilon^2_k/\eta$$

as follows from (2.11) and (3.2). Thus

$$\|\hat{C}^{(k)}\|_F \leqq \frac{42}{83}\ (1+(n-k)^{-1}/2)\varepsilon^2_k/\eta\ .$$

In a similar way one gets, now with (2.9):

$$\delta_k(B^{(k)}P^{(k)}) \leqq \|\hat{B}^{(k)}\|_F \|H^{(k)}\|_2 \leqq \frac{42}{83} (1+(n-k)^{-1}/4)(1+(n-k)^{-2}/100)\ \varepsilon_k^2/\eta$$

$$\leqq \frac{42}{83} (1+3(n-k)^{-1}/8)\ \varepsilon_k^2/\eta\ .$$

Hence

$$\|\hat{G}^{(k)}\|_F \leqq \frac{84}{83} (1+7(n-k)^{-1}/10)\ \varepsilon_k^2/\eta\ . \qquad \square$$

THEOREM 3.5. If $\varepsilon_0/\eta \leqq (10n)^{-1}$ then

$$\varepsilon_{n-1} \leqq \frac{54}{53} \sum_{k=1}^{n-2} (1+\frac{7}{10}(n-k)^{-1}) \prod_{j=0}^{k-1} (1+(n-j)^{-1}/5)^2\ \varepsilon_0^2/\eta\ . \tag{3.17}$$

PROOF. By (3.15) and (3.16)

$$\varepsilon_{n-1} \leqq \frac{155}{154}\frac{84}{83} \sum_{k=1}^{n-2} (1+\frac{7}{10}(n-k)^{-1})\ \varepsilon_k^2/\eta\ . \tag{3.18}$$

According to theorem 3.1 $\varepsilon_k/\eta \leqq 1/10$ for $k = 0,1,\ldots, n-1$ if $\varepsilon_0/\eta \leqq \frac{1}{10}\eta^{-1}$. Hence by (3.1) and (3.2),

$$\varepsilon_{k+1} \leqq (1+(n-k)^{-1}/5)\varepsilon_k, \quad k = 0,\ldots, n-2$$

Thus

$$\varepsilon_k \leqq \prod_{j=0}^{k-1} (1+(n-j)^{-1}/5)\ \varepsilon_0\ , \quad k = 0,\ldots, n-1\ . \tag{3.19}$$

So (3.17) is a consequence of (3.18) and (3.19). $\square$

LEMMA 3.6. Let be

$$t_n = \sum_{k=1}^{n-2} (1+\frac{7}{10}(n-k)^{-1}) \prod_{j=0}^{k-1} (1+(n-j)^{-1}/5)^2$$

Then for $n \geqq 3$

$$t_n \leqq 11\, n/6 + 1$$

PROOF. It is easy to see that t_n satisfies the recurrence relation

$$t_{n+1} = \left(1+ \frac{1}{5}\, (n+1)^{-1}\right)^2 (t_n+1+7n^{-1}/5)\ .$$

With starting value $t_3 = \frac{27}{20}\, (16/15)^2$ one verifies that

$t_n \leqq \frac{11}{6}\, n + 1$, $n = 3,4,\dots, 9$. Now assume $t_n \leqq 11\, n/6 + 1$. Then

$$t_{n+1} \leqq \left(1+ \frac{1}{5}\, (n+1)^{-1}\right)^2 \left(\frac{11}{6}\, n+2+7n^{-1}/5\right) \leqq 11(n+1)/6 + 1$$

for $n \geqq 9$. □

This lemma together with theorem 3.6 enables us to finish the proof of the quadratic convergence as formulated in

THEOREM 3.7. If $\varepsilon_0/\eta \leqq (10n)^{-1}$ then

$$\varepsilon_{n-1} \leqq c_n\, \varepsilon_0^2/\eta \tag{3.20}$$

where $c_n = \frac{54}{53}\, (11n/6+1)$. So

$$\varepsilon_{k(n-1)} \leqq (c_n/\eta)^{2^k-1}\, \varepsilon_0^{2^k}\ .$$

PROOF. (3.17) and (3.18) give

$$\varepsilon_{n-1} \leqq \frac{54}{53}\, (11n/6+1)\, \varepsilon_0^2/\eta < \frac{1}{5}\, \varepsilon_0$$

since $\varepsilon_0/\eta \leqq (10n)^{-1}$. This guarantees the quadratic convergence. □

4. CONCLUSIONS

It goes without saying that our proof shares important features with the proof of the analoguous problem for real symmetric matrices [6,7,11,12]. As in the paper of Ruhe [9] the complexity of the proof rises from the non-unitary transformations. These hinder the monotonic decrease of the norm of the non-diagonal parts $E^{(k)}$ of the successive matrices $A^{(k)}$.
The proof is restricted to the case of almost diagonal matrices in the sense that $\|E^{(0)}\|_F$ is small relatively $\eta = \min_{i \neq j} |\lambda_i - \lambda_j|$, the separation of A's spectrum.
The method clarifies that in the final stage of a norm reducing eigenvalue algorithm the annihilation procedure suffices for the convergence of the process and can be executed in stead of the complicated and time consuming normreducing shear transforms in both the Eberlein as in the Sameh algorithm [3,8,14].
The long sequence of worst case estimates brings about that despite their stepwise sharpness even the final bound (3.20) underestimates the convergence quite considerably. The uniform lowerbound $\eta(1-2\varepsilon_k/\eta) \leqq \vartheta_k$ for all $|a_{ii}^{(k)} - a_{jj}^{(k)}|$ especially leads to a pessimistic result.
A numerical result illustrates that the condition $\varepsilon_0/\eta \leqq (10n)^{-1}$ is far too strong as appeared in many test cases. The matrix is of order 6. A = D + E with

$$D = \mathrm{diag}(1,36,29,22,8,15), \; e_{ij} = 2.8(j-i) \; . \tag{4.1}$$

The eigenvalues of A are 3.416218, 9.792723, 15.704854, 21.295146, 27.207277 and 33.583782. So η = 5.590292.

k	ε_k	k	ε_k	k	ε_k
0	<u>15.336223</u>	7	3.647763	14	$0.973630.10^{-3}$
1	22.305149	8	2.061459	15	<u>$0.624204.10^{-4}$</u>
2	19.654710	9	0.698883	16	$0.631612.10^{-5}$
3	15.030077	10	<u>0.257238</u>	17	$0.816325.10^{-7}$
4	11.300451	11	$0.907285.10^{-1}$	18	$0.544994.10^{-8}$
5	<u>5.509465</u>	12	$0.595277.10^{-1}$	19	$0.476108.10^{-10}$
6	4.941476	13	$0.470223.10^{-2}$	20	<u>$0.473872.10^{-15}$</u>

Fig. 4. The sequence ε_k with caterpillar cyclic order.

Figure 4 illustrates that the convergence occurs despite the large ε_0/η. There rest interesting researchproblems. Firstly how to modify the algorithm in case of multiple eigenvalues. The papers [4,13] give useful information for the structure of an almost diagonal matrix with multiple eigenvalues. It seems reasonable to skip the shear transform T_{ik} if the corresponding diagonal elements of $A^{(k)}$ are affiliated with equal eigenvalues.

The criterion, based on $\min_{i \neq j} |a_{i,i}^{(k)} - a_{j,j}^{(k)}|$ and ε_k, for the transfer from the normreduction in the Sameh process to the annihilation procedure is an important question for future research [14].

5. REFERENCES

1. Bauer, F.L. and C.T. Fike, Norms and Exclusion Theorems, Num. Math., 2, 1960, pp. 137-141.

2. Brent, R.P. and F.L. Luk, The Solution of Singular-value and Eigenvalue Problems on Multiprocessor Arrays, SIAM J. Sci.Stat. Comput., 6, 1985, pp. 69-84.

3. Eberlein, P.J., A Jacobi-like Method for the Automatic Computation if Eigenvalues and Eigenvectors of an Arbitrary Matrix, J. Soc. Indust. Appl. Math., 10, 1962, pp. 74-88.

4. Fan, K. and A.J. Hoffman, Lower Bounds for the Rank and Location of the Eigenvalues of a Matrix. Nat. Bur. Stand. Appl. Math. Series 39, 1954, pp. 117-130.

5. Jacobi, C.G.J., Über ein leichtes Verfahren die in der Theorie der Seculär Störungen vorkommenden Gleichungen numerisch aufzulösen, Crelle's J., 30, pp. 51-96.

6. Kempen, H.P.M. van, On the Convergence of the Classical Jacobi Method for Real Symmetric Matrices with Nondistinct Eigenvalues, Num. Math., 9, 1966, pp. 11-18.

7. Kempen. H.P.M. van, On the Quadratic Convergence of the Serial Cyclic Jacobi Method, Num. Math., 9, 19-22.

8. Paardekooper, M.H.C., An Eigenvalue Algorithm Based on Normrreducing Transformations, Technische Universiteit Eindhoven, Thesis, 1969.

9. Ruhe, A., On the Quadratic Convergence of a Generalization of the Jacobi Method for Arbitrary Matrices, BIT, 8, 1968, pp. 210-233.

10. Sameh, A.H., On Jacobi and Jacobi-like Algorithms for a Parallel Computer. Math. Comp., 25, 1971, pp. 579-590.

11. Schönhage, A., On the Quadratic Convergence of the Jacobi Process, Num. Math., 6, 1964, pp. 410-412.

12. Wilkinson, J.H., Note on the Quadratic Convergence of the Cyclic Jacobi Process, Num. Math., 4, 1962, pp. 296-300.

13. Wilkinson, J.H., Almost Diagonal Matrices with Multiple or Close Eigenvalues, Lin. Alg. Appl., 1, 1968, pp. 1-12.

14. Paardekooper, M.H.C., Sameh's parallel eigenvalue algorithm revisited, in *Algorithms and Applications of Vector and Parallel Computers*, (H.J.J. te Riele, T.H.J. Dekker and H.A. van der Vorst, Eds.), Elsevier Science Publishers, 1987, pp. 351-371.

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